PROPOSITIONES
QUÆDAM
GEOMETRICÆ
SOLUTÆ & DEMONSTRATÆ.

QUAS
Consent. Ampl. Facult. Philos. in Reg. Acad. Aboënsi
Publico examini submittunt

AUCTOR
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ET
RESPONDENS
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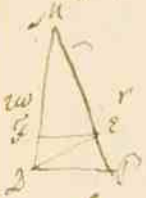
In AUDITORIO MAJORI, Die XXXI Maji
An. MDCCLXXIV.
Horis. A. M. Consvetis.

ABOÆ,
Typis JOHANNIS CHRISTOPHORI FRENCKELL.

3.

$\frac{dr}{r} = \frac{\;}{\alpha} \ldots$

$r\,\mathrm{Cos}\,\varphi^3 = \ell N^{\frac{s}{\alpha}} = 2\omega\,\mathrm{Cos}\,\varphi^2 = \frac{8\omega^3}{r^2}$



$r : 2\omega : 2\omega : 2\omega\,\mathrm{Cos}\,\varphi = \frac{4\omega^2}{r}$

$r : 2\omega :: 2\omega\,\mathrm{Cos}\,\varphi : 2\omega\,\mathrm{Cos}\,\varphi^2 = \frac{8\omega^3}{r^2}$

$\frac{ds}{\alpha} = 3\frac{d\omega}{\omega} - 2\frac{dr}{r}$; $tg\,\varphi = \frac{\sqrt{r^2-4\omega^2}}{2\omega}$

$\cancel{\frac{dr}{3\varphi} = \left(3\frac{d\omega}{\omega} - 2\frac{dr}{r}\right)\left(\frac{r}{\alpha} + 3\sqrt{r^2}\right)}$

$ds = \alpha\left(3\frac{d\omega}{\omega} - 2\frac{dr}{r}\right)$; $-dr = ds\left(\frac{r}{\alpha} + 3\frac{\sqrt{r^2-4\omega^2}}{2\omega}\right)$

$-dr = \left(3\frac{d\omega}{\omega} - 2\frac{dr}{r}\right)\left(r + 3a\frac{\sqrt{r^2-4\omega^2}}{2\omega}\right)$

$+dr = 3\frac{rd\omega}{\omega} + \frac{9ad\omega\sqrt{r^2-4\omega^2}}{2\omega^2} - \cancel{2dr} - \frac{6adr\sqrt{r^2-4\omega^2}}{2\omega r}$

$dr\frac{(2\omega r + 6a\sqrt{r^2-4\omega^2})}{\cancel{2\omega} r} = \frac{6\omega r + 9a\sqrt{r^2-4\omega^2}}{2\omega}d\omega$

$dr(2\omega^2 r + 6a\omega\sqrt{r^2-4\omega^2}) = d\omega(6\omega r^2 + 9ar\sqrt{r^2-4\omega^2})$

$6\omega r + 9a\sqrt{r^2 \cancel{+} 4\omega^2} \;\Big|\; 2\omega r + 6a\sqrt{r^2-4\omega^2}$, 3

$-6\omega r - 18a\sqrt{\;}$

$-9a$

$2\omega = \frac{ds^2}{gdt^2}$; $\mathrm{Cos}\,\varphi^2 = \frac{g\ell N^{\frac{s}{\alpha}}dt^2}{ds^2}$

$\mathrm{Cos}\,\varphi = \frac{AN^{\frac{s}{2\alpha}}dt}{ds}$ $\quad \frac{2\omega}{\mathrm{Cos}\,\varphi} = r = -\frac{ds}{d\varphi} = +\frac{ds}{BN^{\frac{s}{2a}}dt}$

$-d\varphi = BN^{\frac{s}{2a}}dt$; $d^2t = 0$; $dr = \frac{d^2s}{BN^{\frac{s}{2a}}dt} - \frac{ds^2}{aBN^{\frac{s}{2a}}dt}$

$-\frac{dr}{ds} = \frac{ds}{aBN^{\frac{s}{2a}}dt} - \frac{d^2s}{BN^{\frac{s}{2a}}dt\,ds} = \frac{ds}{aBN^{\frac{s}{2a}}dt} + \frac{3\sqrt{1 - \frac{A^2N^{\frac{s}{a}}dt^2}{ds^2}}}{\frac{AN^{\frac{s}{2a}}dt}{ds}}$

$\frac{d^2s^2}{\cancel{BN^{\frac{s}{a}}dt^2ds^2}} = \frac{9(ds^2 - A^2N^{\frac{s}{a}}dt^2)}{A^2N^{\frac{s}{a}}dt^2}$ $\quad ds = pdt$, $d^2s = dp\,dt$

$\frac{d^2s^2}{ds^2} = \ell ds^2 - AN^{\frac{s}{a}}dt^2$; $\frac{dp^2}{p^2} = \ell ds^2 - \frac{AN^{\frac{s}{a}}ds^2}{p^2}$

$dp^2 = \ell p^2 ds^2 - AN^{\frac{s}{a}}ds^2$; $p = N^{\alpha s}z$

$dp = N^{\alpha s}(\alpha z\,ds + dz)$

Post Euclidis & Archimedis tempora clariss.mi extiterunt Mathem. Apollonius Pergaeus (Conicor. lib. 7.), Hipparchus & Menelaus (de subtensis in circulo), Theodosius (Sphaericor. l. 3), Ptolemaeus Astronomorum princeps, Diophantus algebrae inventor & vere totius numericae subtilitatis magister, Proclus, Theon, Pappus &c. &c

I. N. J.

Ab antiquissimis retro temporibus, excultam in primis fuisse Geometriam, magnique a mortalibus æstimatam, multi eorum, qui scientiarum exposuerunt fata, testantur. Cum enim reliquæ disciplinæ aut essent neglectæ penitus, aut multis saltem ænigmatibus involutæ falsisque opinionibus commixtæ, Geometria, quantumvis initio sine dubio rudis atque haud multas easque facillimas complectens veritates, suo tamen non destituta nitore, brevi nacta est cultores haud contemnendos. Magna semper fuit Mathematicorum de Euclide & Archimede opinio; neque unquam horum opera collaudare ac admirari desistent. Optandum autem esset, ut plurium Seculorum eos subsequentium Mathematici, veritates ab iis inventas atque dispositas in usus vertere voluissent suos, neque id satis habuissent si eorum libros exscriberent. Vix namque unus aut alter per multa secula in studia rite incubuit Mathematica; donec tandem seculo decimo sexto, quod faustum omnibus disciplinis, male adhuc habitis, esse cœpit, haud pauci Mathematica cognitione sunt illustrati. Proxime autem præterlapso, viros in omnibus Matheseos partibus celeberrimos vidit orbis eruditus. Penitius enim in hæc arcana penetrarunt, & Geometriam præcipue ſublimiorem excoluerunt, ut plures taceam, Gregorius a S. Vincentio, Renat. Carteſius, Chriſt. Hugenius, Jo. Walliſius ac denique ſagaciſſimus ille Geometra Iſ. Barrowius; maximam autem reliquit ſui famam & virtute & Mathematica cognitione partam, Princeps omnium Mathematicorum Iſaacus Newtonus. Hic, quo pollebat ingenii acumine prorſus incomparabili, difficillima problemata mira ſoluta dedit concinnitate. Nam, præter novum, quod in Optica ſua condidit, de natura lucis ſyſtema, veras naturæ leges diſtincte primus expoſuit, & totius ſyſtematis Mundani explicationem ex theoria gravitatis non minus ingenioſe quam ſolide deduxit. Quæ autem omnia Methodo Fluxionum a ſe inventæ ſuperſtructa ſynthetice demonſtravit in ſuis *Philoſophiæ Naturalis Principiis Mathematicis*. Quibus magni Newtoni inventis novam induit faciem quantitatum ſcientia, atque alia adminicula varia ad eam locupletandam cum acceſſiſſent, intra ſat breve tempus incrementa cepit prorſus admiranda. Quo etiam factum eſt, ut ætas noſtra nobis præſtiterit Geometras ac Mathematicos præſtantiſſimos. Præcipuam enim rerum Mathematicarum habent laudem Viri Excel. Euler, Simſon, Clairaut, d' Alembert, Klingenſtierna, aliique. Quorum opera ac induſtria quamquam Matheſis ad tantum faſtigium jam eſt evecta, quantum forte ſperare vix potuerat antiquitas; novæ tamen in dies ſingulos veritatum fieri poſſunt acceſſiones, & quanto quis eſt Mathematice doctior, tanto plura etiamnum invenienda reſtare intelligit. Nec mortalibus, quorum omnes ingenii vires valde ſunt coarctatæ, datum eſt omnia perſpicere & perſcrutari. Dici non poteſt, quantum nos lateat, & plurima forte occulta ſemper manebunt. Cujus rei probandæ documenta, in ſcientiis quoque Mathematicis valde luculenta occurrunt. Sunt enim res ſat

A

multæ,

$ds = p\,dt$; $dp^2 = C p^2 ds^2 + A N^{\frac{s}{a}} ds^2$; $p = N^{\frac{s}{2a}} z$;

$dp = N^{\frac{s}{2a}} \left(\frac{z\,ds}{2a} + dz \right)$

$\frac{z^2 ds^2}{4a^2} + \frac{z\,dz\,ds}{a} + dz^2 = C z^2 ds^2 + A ds^2$; $C - \frac{1}{4a^2} = \alpha^2$

$\cancel{z\,dz\,ds}$ $\alpha^2 z^2 ds^2 + \frac{z}{a} dz\,ds + dz^2 = A^2 ds^2$

$$dp = ds \sqrt{C p^2 + A^2 N^{\frac{s}{a}}} = N^{\frac{s}{2a}} ds \sqrt{C z^2 + A^2}$$

$$= N^{\frac{s}{2a}} \left(\frac{z\,ds}{2a} + dz \right)$$

$$2a\,dz = ds \left(2a \sqrt{C z^2 + A^2} - z \right)$$

$$ds = \frac{2a\,dz}{2a \sqrt{C z^2 + A^2} - z};$$

multæ, Mathematicam quidem conſiderationem haud reſpuentes, quarum tamen cognitionem vel prorſus nullam vel admodum mancam eſſe & imperfectam deprehendimus. Quid! quod quædam eorum, quæ in quibusdam Matheſeos adplicatæ partibus cognita jam ſatis atque perſpecta quidam putarunt, variis laborant defectibus, atque ſic multo accuratiorem requirunt diſquiſitionem. Itaque ſcientiis omnibus, quibus Matheſis uſui eſſe poteſt, optime ut conſulatur, judicio & exemplo præſtantiſſimorum Geometrarum, Geometria tanto magis eſt excolenda, quanto certius eſt eam utramque in Matheſi adplicata facere paginam. Sic multas veritates, quarum hodie vel aliquantam tantum vel prorſus nullam habemus notitiam, poſteritatis induſtria ſine dubio deteget. Qua de re convictus, eorum quibus vel tantillum Mathemata tractare incumbit, putavi eſſe officium, ut pro virili parte in his ſtudiis ſe exerceant. Neque igitur ego meo deeſſe volui officio. Variis autem multisque occupationibus impedito, non niſi faciliora tentare licuit; adeoque non quidem ampla Matheſeos ſupellectilem augentia hac in opella reperiuntur; quædam tamen vel propria meditatione elaborata, vel ab aliis inventa noſtris demonſtrationibus firmata ac dilucidata, exhibemus. Quapropter B. L. eo mitiorem in partem hæc conata te ſpero interpretaturum, quo certius eſt, morem eſſe Geometrarum, cum ſublimia exſpectant, neque ſpernere leviora. De cætero voti me damnatum putabo, ſi quid tyronibus, quibus Mathemata ſunt in deliciis, hæc opella mea delectationis adferat, eosque ad ſuas in hoc ſtudiorum genere periclitandas ingenii vires exſtimulet.

PROPOSITIO I. THEOREMA. Fig. 1. 2. 3.

Si per singulos vertices Trianguli ABC (plani & scaleni) circulo inscripti, agantur rectæ circulum tangentes AD, BE, CF, quæ lateribus Δ:li oppositis BC, AC, AB occurrant, singulæ singulis, in D, E, F; erunt tria hæc puncta occursus in recta linea (*).

Dem. Per E, C, ducantur rectæ EH, CQ, parallelæ ipsis BC, BA. Et quoniam (per. pr. 29. libr. I. Eucl.) ang. ABC + BCQ = 2 rectis, atque (Fig. 1. 3.) (pr. 16. l. I.) ang. ABC > BDA, erunt ang. ADC + DCQ < 2 Rectis; consequenter recta DA, producta si opus fuerit, productæ CQ occurret in Q. Occurrat etiam recta EH rectis BA, CF, productis in L & H. Patet igitur (pr. 4. l. VI.) esse DB : DC :: BA : CQ. Cum vero sit (pr. 32. l. III.) ang. CAQ = ABC & (pr. 29. l. I.) ang BAC = ACQ, erit Triang. BAC ∽ Δ:lo ACQ, ideoque BA : AC :: AC : CQ & (pr. 20. l. VI.) BA : CQ :: BAq : ACq, quare DB : DC :: BAq : ACq. Porro quoniam (pr. 29. l. I. & 32. l. III.) ang. EBL = BCA = HEC = (dem) AQC, & ang. ELB = ABC = ECH = CAQ, erunt Triangula EBL, BAC, ECH & ACQ æquiangula & similia. Ex jam dictis facile quoque intelligitur, esse HC = EB. Est itaque EL : EB (= HC) :: BA : AC, & HC : HE :: BA : AC, atque per compositionem rationis, EL : HE :: [BAq : ACq ::] DB : DC, & denique invertendo, dividendo, alter-

(*) Hoc Theorema, quod elegantem quandam circuli proprietatem sistit, & a Nobil. D:no KLINGENSTIERNA inventum perhibetur, benigne mecum, dum vixit, communicavit Cel. D:us Prof. M. J. WALLENIUS; qui quidem, quod mihi notum, ejus quoque dedit demonstrationem, ab illis tamen, quæ hic proferuntur, diversam.

alternando, EL : DB :: [LH : BC ::] FL : FB (*). Quam ob rem (Cfr. pr. 4. & 6. l. VI.) puncta D, E, F sunt in linea recta. Q. E. D.

Aliter. Duc rectam FK parallelam ipsi BE, & junge AK. Jam (Fig. 1. 2.) ob ang. BFK = EBF = BCA, & angulum FBK communem, vel (Fig. 3.) ABC = KBF & KFB = EBA = BCA, erunt Triangula FBK, BAC æquiangula & similia, ideoque BF : BK :: BC : BA & (pr. 16. l. VI.) BF. BA = BK. BC. Consequenter, ut facile intelligitur ex prop. 36. & 35. l. III. Eucl. erunt F, A, K, C in peripheria circuli per hæc puncta transeuntis. Hinc ergo sequitur (pr. 21. l. III.) ut sit (Fig. 1. 2.) ang. AKF = ACE (= ABC = QAC =) EAM (pr. 15. l. I.) vel (Fig. 3.) AKF = (ACH = ABC = QAC =) EAM. Sed, ob parallelas FK, EB, & quia FK occurrit rectæ DQ productæ, si opus sit, in R, erit ang. EMA = ARK; ergo & reliquus ang. MEA = RAK & Triang. EMA æquiangulum & simile Δ:lo AKR; ideoque KR : RA :: AM : ME. Porro, quoniam ang. AFR (= MBA = BCA = MAB) = FAR, erit (pr. 6. l. I.) MB = MA & AR = RF, ideoque KR : RF :: BM : ME, & invertendo FR : RK :: EM : MB, componendo FK : RK :: EB : MB, ac denique alternando FK : EB [:: RK : MB] :: DK : DB (**). At FK, EB parallelæ; ergo.

 Ali-

(*) In eo casu quo punctum H (Fig. 2.) cadit inter F & L, si jungantur EF, FD, erit ob ang. ELB = LBD & EL : DB :: FL : FB, Triang. ELF ∽ Δ:o BFD, ideoque ang. EFL = BFD. Sed BFL est una recta; igitur EF, FD sunt in directum (pr. 14. l. I.)

(**) Quod jam supra in nota ad præcedentem demonstrationem diximus, hic etiam pro casu 3:io (Fig. 3.) notandum.

Aliter. Agatur, ut antea, ipsi BE parallela recta FK occurrens lateri BC producto, si opus fuerit, in K puncto. Jungantur puncta A, K recta KA, quæ producta occurrat rectæ BE productæ, quando necesse est, in puncto N. Tum, ut facile apparet, erit Triang BNA ∽ Δ:o FAK ideoque BN : NA :: FK : KA. Similiter facili negotio intelligitur, esse Triang. NMA ∽ Δ o RAK & AN : NM :: AK : KR; Adeoque ex æquo BN : NM :: FK : KR. Porro, cum per prius demonstrata, Triang. MEA ∽ Δ:lo AKR ∽ Δ:lo MAN, erit NM : MA :: MA : ME, vel, ob MA = MB, NM : MB :: MB : ME, & invertendo, componendo, BN : NM :: BE : BM. adeoque BE : BM :: FK : KR, & alternando BE : KF :: [BM : KR ::] DB : DK. Q. E. D.

Hinc facile solvitur sequens

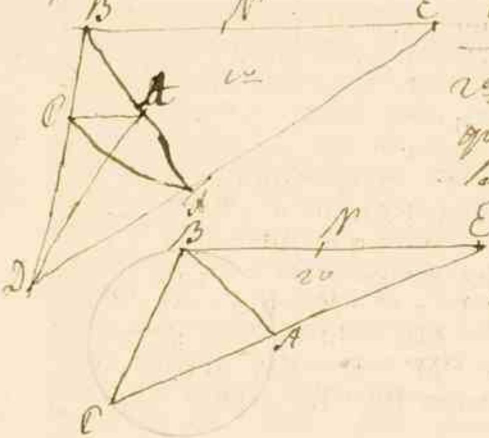
PROBLEMA. PROPOSITIO. II. Fig. 1. 2. 3.

Datis in recta linea tribus punctis N, E, B, tale construere Triangulum BAC, cujus unius anguli vertex erit in B, ut per reliquorum angulorum vertices A & C duci possint rectæ CF, AD, lateribus Trianguli oppositis BA, BC occurrentes in punctis F & D, quæ erunt in eadem recta cum uno punctorum datorum E.

Sol. Cum data sint puncta B, N, E, dantur rectæ NB, EB. Sumatur igitur ipsis NB + EB, NB & EB quarta proportionalis BM; Tunc semper cadet punctum M inter E & B, quia NB + EB > NB, adeoque etiam EB > BM. Centro M, intervallo MB, describe semicirculum BAG, cujus in peripheria sume punctum quodcunque A. Junge puncta NA, EA, MA & BA. Atque sic datur angulus EAM, cui æqualem constitue ad punctum B & rectam BA, angulum ABC. Produc rectam

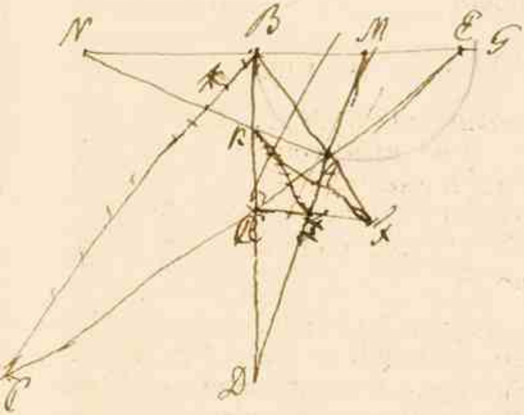
EA

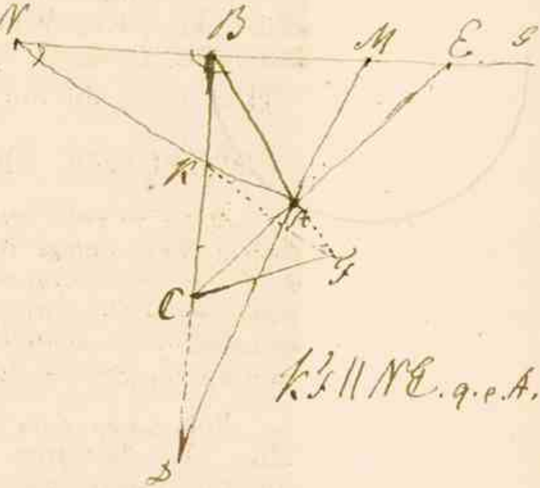
Prop. 2. Problematis ~~...~~ hujus solutio quam tradidit non est generalis nec satis concinna. Sic enim generaliter & simul simplicissime solvi potest: duc utramque [producta] BC & BA & ED &c.



1°. describe circulum quemcunque qui tangat rectam BE in B. duc BC utrimque &c

3°.





KF || NE. q.e.A.

EA donec occurrat rectæ BC in C; quod semper fieri debet, quoniam ang. BAC + BAE = 2 Rectis & EAB > (EAM =) ABC, ideoque BAC + ABC < 2 Rectis.‡ +. ACH = EAM.
Ad punctum C quod jam datur, fiat angulus†ACF = EAM (vel = (Fig. 3.) hujus complemento ad duos rectos MAC.) Producantur rectæ CF, AM, donec occurrant lateribus Trianguli oppositis & productis BA, BC, in F & D. Producta quoque recta NA ad K, junge puncta KF & protrahe rectam DA, usque dum attingat rectam FK in puncto R. Denique (pr. 5. l. IV.) circa Triangulum sic constructum BAC circumscribe circulum. Dico factum. Nam per constructionem est NB + EB : NB :: EB : BM, & dividendo EB : NB :: EM : BM, alternando EB : EM :: NB : BM, ac iterum dividendo BM : ME :: NM : BM. Sed BM = MA, quia sunt radii circuli BAG; adeoque NM : MA :: MA : ME; consequenter (pr. 6. l. VI.) $\triangle$:um MNA $\sim$ $\triangle$:lo MEA & ang. MEA = MAN, MAE = MNA. Jam vero (pr. 32. l. I.) ang. EAM + MAB = (EAB =) ABC + BCA. Sed (constr.) EAM = ABC, quare erit MAB = ACB. Cum autem per A punctum in peripheria circuli CAB ducta sit recta AD sic, ut faciat angulum BAD æqualem illi ACB, qui in alterno circuli segmento BCA consistit, facile (pr. 32. l. III.) intelligitur, rectam AD contingere circulum CBA in puncto A. Porro, quia BM = MA, atque sic ang. MBA = (MAB =) BCA, continget ob eandem rationem recta BE circulum in B. Pariter, quoniam (constr.) ang. ACF = MAE = ABC vel (Fig. 3.) ACH = ABC, continget etiam recta CF circulum CBA in puncto C. Sub data igitur hypothesi, constructum est Triangulum BAC atque per vertices angulorum ejus ductæ rectæ CF, AD, circulum circumscriptum tangentes, & lateribus Trianguli oppositis & productis BA, BC occurrentes in punctis F & D. Contingit quoque recta data BE

† scil. ad alteram partem rectæ AC respectu puncti B.

‡ 1° circumscribe circulum & 2° duc tangentes &c

BE circulum circumscriptum in B, atque occurrit lateri Trianguli opposito AC in E. Quare (per Theorema nuper demonstratum) puncta F & D erunt in eadem linea recta cum puncto E.

Sol. 2. Ex dictis adparet, esse $\triangle$ EBA $\backsim$ $\triangle$:lo EBC, ideoque AE : EB :: EB : EC. Assumto igitur, ut antea, puncto A in peripheria circuli jam inventi BAG, haud difficulter determinatur Triangulum BAC, capiendo, in recta EA producta, EC = tertiae proportionali ad EA & EB, quae dantur positione & magnitudine. Junge puncta B, C recta BC; atque sic inventum est Triangulum BAC. Datur quoque N punctum, adeoque etiam recta NA, quae producta occurret lateri BC, producto si & quoad opus fuerit, in K puncto, unde parallela ipsi BE agatur recta KF, donec attingat rectam BA productam, in puncto F. Jungantur puncta F, C & producatur AM ad D. Quo facto, erunt F & D puncta quaesita. Etenim, quia AE : EB :: EB : EC (constr.) & ang. BEA communis, erit (pr. 6. l. VI.) $\triangle$ BEC $\backsim$ $\triangle$:lo EBA, & ang. EBC = EAB, ang. ECB = EBA = MAB. Quare rectae BE, AM contingent in punctis B & A circulum circa Triangulum BAC circumscriptum. Porro, cum sit KF parallela ipsi BN, erit (Fig. 1. & 3.) ang. AFK = NBA = (p. prius) KCA, vel (Fig. 2.) ang. CKF + CAF = CKF + KBE = 2 Rectis. Ideoque puncta F, A, K, C erunt in peripheria circuli & (Fig. 1. 2.) ang. ACF = AKF = (ob paral. KF, BN) BNA vel (Fig. 3.) ang. ACF + AKF = 2 Rectis; consequenter ang. ACH = AKF = BNA. Verum (pr. 32. l. I.) ang. EAM + MAB (= EAB) = ABC + ACB, ideoque, ob ang. MAB = ACB, est ang. EAM = ABC. Praeterea, cum (prius dem.) sit ang. BNA = EAM, erit (Fig. 1. 2.) ang. ACF = ABC vel (Fig. 3.) ACH = ABC. ideoque continget etiam recta CF in puncto C cir-

circulum circa Triangulum CBA circumſcriptum; unde cætera intelliguntur, ut in demonſtratione ſolutionis præcedentis oſtenſum.

Cor. Punctorum N & E a ſingulis peripheriæ BAG punctis diſtantiæ erunt in ratione conſtante ſcilicet in ratione ſubduplicata diſtantiarum NM & EM a centro M, vel in ratione ſimplici diſtantiarum NB : EB. Nam oſtendimus eſſe $\triangle$ NMA $\sim$ $\triangle$:lo MEA, ideoque NM : MA :: MA : ME & (pr. 20. l. VI.) NMq : MAq :: NM : ME. Eſt autem, propter ſimilitudinem eorundem Triangulorum, NA : NM :: AE : AM, adeoque NA : AE :: (NM : AM ::) $\sqrt{NM} : \sqrt{ME}$. Porro, ob $\triangle$:um EBA $\sim$ $\triangle$:lo EBC erit EB : BC :: EA : AB; & quia (dem) ang. NBA = ACB & ang. ABC = EAM = MNA, erit $\triangle$ BNA $\sim$ $\triangle$:lo BAC & BC : BA :: BN : NA, quare per compoſitionem rationis, EB : BA :: EA $\times$ BN : BA $\times$ NA, & conſequenter (pr. 16. l. VI.) EB $\times$ NA = EA $\times$ BN & NA : EA :: NB : EB. Q. E. A.

Schol. 1. Si ang. AMB + MBC > 2 Rectis, concurrent rectæ AM & CB productæ ſic, ut punctum B cadat inter D & C, (Fig. 1. 3.). Si AMB + MBC < 2 Rectis, cadit C inter B & D, (Fig. 2.). Poſitis autem angulis AMB + MBC = 2 R, erit ang. EMA = MBC = MBA + ABC = EAM + MAB = MBA + MAB, ideoque ang. MBA = EAM = MNA & BA = AN. Hinc in peripheria circuli BAG determinari poteſt punctum α, ad quod ſi ducatur recta Bα & cætera conſtruantur ut prius, erit Mα parallela ipſi BC & Triangulum conſtruendum fiet Iſoſceles, quod Problemati non ſatisfacit. Biſecetur nimirum (Fig. 1.) BN in π, unde erigatur rectæ perpendicularis $\pi\alpha$ occurrens peripheriæ Circuli BAG in puncto quodam α, quod erit punctum quæſitum. Hoc facile oſtendi poteſt.

Schol. 2. Prout $MB > = < ME$, fit angulus BAC, in Triangulo construendo, obtusus, rectus vel acutus. Si enim ducta intelligatur recta GA, erit in casu primo angulus in semicirculo GAB $>$ ang. EAB, consequenter ang. EAB acutus & ejus complementum ad duos rectos BAC obtusus. In casu tertio erit ang. GAB $<$ EAB, hoc est, ang. EAB obtusus ejusque complementum BAC acutus. Si autem $BM = ME$, facile patet, esse ang. BAC rectum.

Potest ee $AB > AC > BC$
~~Ergo~~ & $AB = AC = BC$
Ergo.

Schol. 3. In Triangulo sic constructo BAC erit semper basis BC major latere BA. Si enim $BC = <$ BA, foret ang. $BAC = < ACB$ vel $= < MBA$. Addito utrinque ang. BAE, erunt ang. $BAC + BAE = < MBA + BAE$. Sed (pr. 32. l. I.) ang. $MBA + BAE <$ 2 rectis, ideoque ang. $BAC + BAE <$ 2 rectis, quod est absurdum. (pr. 13. l. I.). Potest autem esse $AC < = > BC$. Ubi tamen notandum, AC non esse $= > BC$, nisi $MB > MN$, vel N cadat intra circulum BAG. Nam, ob $\triangle BAC \backsim \triangle NBA$, est $BC : AC :: NB : BA$; Quam ob rem, si $AC = > BC$, erit $BA = > NB$, h. e. $MB > MN$, & quidem, si $BA > BN$, cadet F punctum ad partem alteram rectæ BC (Fig. 3).

Non potest ee $AC = BC$ nec $AC = AB$, sic enim non foret $\triangle$ scalenum.

Fieri etiam potest, ut sit $AC > = < AB$. Dum A punctum est (Fig. 1.) inter G & α erit $AC > AB$; quia in eo casu (cfr. Schol. 1.) $BA > AN$ adeoque ang. $BNA > ABN$. At ang. $BNA = MAE = ABC$ & $ABN = ACB$, quare ang. $ABC > ACB$ & $AC > AB$. Quando autem A in α cadit, diximus (Schol. 1.) Triangulum fore isosceles & $BA = AC$. Si vero cadat A inter B & α erit $AC < AB$. Quod similiter facile ostendi potest.

Schol. 4. Quæritur quomodo construi oporteat Triangulum BAC (Fig. 1.) ut fiat ang. $ABC = 2\,ACB$? Est igitur in data hypothesi determinandum punctum A, quod fieri sic potest. Seca bifariam rectam MN in O. Ex hoc

pun-

puncto duc rectam perpendicularem OA, quæ secabit circumferentiam circuli BAG in puncto quæsito A. Etenim, quia (per constr.) MO = ON, OA Latus commune & ang. MOA = AON, erit ang. AMN = MNA = (p. prius dem.) MAE = ABC. Sed AMN = MBA + MAB = (dem.) 2 MBA = 2 ACB. Ergo ang. ABC = 2 ACB. Idem etiam obtinetur, si centro E intervallo EM describatur circulus, qui secet circumferentiam circuli BAG in puncto A.

Prius vero quam inveniri possit Triangulum BAC, cujus angulus ACB sit duplus anguli ABC, construendum est Triangulum HGL (Fig. 4.), cujus latera GL, HG erunt in data ratione ME : MA & angulus HGL ad verticem quadruplus anguli GHL ad basin HL (*). Quo facto constitue ad rectam EM & punctum M ang. EMA′ = ang.

(*) Hoc Problema sic solvitur: Sit data illa ratio ME : MA :: d : e. Pone rectas LE = d, & ED = e in directum. In LD producta cape DT quartam proportionalem ipsis LE, ED, LD. Super diametro ET describe semicirculum TKE, qui in K secet rectam DK ipsi ET in D perpendicularem. Seca bifariam rectam DE in N & sume NH = NK. Deinde centro D intervallo DH & centro E radio ED describe circulos, qui sese intersecent in aliquo puncto G. Junge jam puncta GH, GL, GE, & erit HGL (vel GEL) Triangulum desideratum. *Dem.* Ducta GD bisecetur angulus DGE recta GR. Dein agatur recta RM parallela ipsi HG, & jungantur puncta M, E. Jam erit (pr. 14. l. II.) TD × DE = DKq = KNq — DNq = (constr.) NHq — DNq = (cfr. pr. 5. l. II.) $\overline{NH + DN} \times \overline{NH - DN}$ = HE × HD. Sed TD × DE : DEq :: TD : DE :: (constr.) LD : LE :: e + d : d. Qua-ang. HGL invento, & erit (per pr. 6. l. VI.) ang. MA'E = GHL, atque ang. EMA' = 4 MA'E. Consequenter ang. A'C'B = MBA' = 2 MA'E = 2 A'BC'. Q. E. F.

Schol. 5. Posito EB = NB, fit, per constructionem, 2 EB : EB :: EB :BM = $\frac{1}{2}$ EB. Hoc in casu, quia angulus in semicirculo EAB rectus, erit quoque angulus deinceps positus BAC rectus. α punctum, ad quod Triangulum fit æquicrurum & MA parallela ipsi BC, situm nunc est in medio peripheriæ circuli BAG. Est

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Quare HE × HD : DEq :: e + d : d. Porro, quia per constructionem, DE = EG & DG = DH, erit ang. EGD = (EDG = DGH + DHG =) 2 DHG. ideoque ang. EGR = EHG & △ EGR ∽ △:lo EGH. Quare HE : EG :: EG : ER & (pr. 16. l. VI.) HE × ER = EGq = DEq. Consequenter HE × HD : HE × ER :: HD : ER :: (e + d : d ::) DL : LE & (pr. 12. l. V.) HL : RL :: DL : LE, ac denique dividendo HR : RL :: DE : EL. At, ob parallelas RM, HG, erit (pr. 2. l. VI.) GM : ML :: (HR : RL ::) DE : EL, ideoque EM parallela ipsi DG, & angulus LEM = LDG = 2 DHG = (ob paral. HG, RM) = 2 MRE. Verum ang. LEM = MRE + EMR, adeoque EMR = (MRE = GHD =) RGE. Ergo puncta E, M, G, R erunt in peripheria ejusdem circuli & ang. EGM = (ERM =) EHG. Unde porro sequitur esse ang. HGL = 4 GHL, ang. HGR = RGL, ang. GEL = HGL, ac denique HG : GL :: (HR : RL :: DE : EL ::) e : d. Q. E. F. Allatam hujus problematis solutionem cum analysi geometrica satis concinna sistit Dissertatio sub Præsidio Cel. D:ni Prof. WALLENII hic Aboæ A:o 1766 edita, cui titulus: Lunulæ quædam circulares quadrabiles. Demonstrationem autem plene expositam & aliquanto mutatam exhibuimus.

Eſt autem punctum O in medio rectæ ME, ideoque arcus GA vel EA (nam puncta E, G, N hoc in caſu coincidunt) ſubtendet chordam EA = AM radio. Quare, ſi centro E radio EM vel MA deſcribatur circulus, ſecabit ille peripheriam BAE in puncto quæſito A; vel, ſi placet, fiat angulus EBA = 30° & punctum, ubi recta BA producta ſecat peripheriam circuli, propoſito ſatis faciet. Similiter, ſi ponatur ang. EBA′ = 60°, determinabitur punctum A′, ad quod, ſi modo ſupra jam oſtenſo conſtruatur Triangulum BA′C′, erit ang. BC′A′ = 2A′BC′; unde ſimul patet eſſe △ BAC ∽ △:lo BA′C′, atque unum angulorum C alterius complementum ad unum rectum. Idem etiam valet de angulo ad B.

Schol. 6. Poſito EB = $\frac{1}{3}$ NB, erit BM = $\frac{1}{3}$ NB & > ME; conſequenter Triangulum conſtruendum obtusangulum (cfr. Schol. 2.). Notandum vero, quod nullum hoc in caſu conſtrui poſſit Triangulum, cujus angulus ad B, nimirum ABC, eſſet duplus anguli ACB. Nam O punctum cadit jam in G & recta perpendicularis ex hoc puncto ducta contingit circulum BAG. Biſecta autem recta BN in π & ducta perpendiculari $\pi\alpha$ erit Triangulum Iſoſceles, cujus angulus obtuſus BαC faciet duas tertias duorum rectorum, quod hic tantum indicaſſe ſufficiat.

Schol. 7. Si EB = $\frac{1}{2}$ NB, adeoque BM = $\frac{1}{2}$ NB, erit Triangulum obtuſangulum & nunquam Iſoſceles; quia punctum π cadit in G & recta hinc ducta perpendicularis contingit circulum. Patet quoque eſſe ang. ACB > ABC. Et in genere eſt obſervandum, ſi EB = < $\frac{1}{3}$ NB nullum conſtrui poſſe Triangulum, cujus angulus ABC eſſet duplus anguli ACB; atque ſi EB = < $\frac{1}{2}$ NB nullum fieri Triangulum æquicrurum, ſed conſtructionem propoſito ſemper ſatisfacere.

Schol. 8. Ponatur jam EB = 2 NB vel NB = $\frac{1}{2}$ EB,

& erit BM = $\frac{2}{3}$ NB = $\frac{1}{3}$ EB. Triangulum BAC poteſt fieri Iſoſceles & habere angulum ABC = 2 ACB. N punctum cadit quoque inter G & M. Et generatim valent hæc omnia, quotieſcunque fuerit EB > NB, adeoque BM > MN. Numquam autem N punctum incidet in M, niſi in puncto B.

PROPOSITIO. III. PROBLEMA. Fig. 5.

Invenire Triangulum NAM a cujus vertice A ſi demittatur recta perpendicularis AE in baſin NM & in hac producta ſumatur MH = ME diſtantiæ videlicet puncti M a recta perpendiculari, atque dein jungantur puncta AH; erit non ſolum unum Trianguli latus AM datæ rectæ MG æquale, verum etiam NA : AM :: NM : AH.

Solutio ſpecialis pro eo caſu ſolum, quo Δ NAM fit rectilineum, qui vero caſus ſ. obtinet niſi pro ~~ME = $\frac{MA}{4}\sqrt{\sqrt{17}-1}$~~ ~~= $\frac{7}{8}$ MA~~ ſ. pro ME minimo, h.e. pro ME = MA$\sqrt{\frac{2}{3}}$,

Sol. A data recta MG abſcinde (pr. 9. l. VI Eucl.) tertiam partem GC. Super MG tanquam diametro deſcribatur ſemicirculus MLG, cui recta perpendicularis CL a puncto C ducta occurrat in puncto L. Centro M intervallo MG deſcribatur circulus GAK, cui occurrat recta ML producta in puncto A, unde agatur ipſi MG ad rectos angulos AE. Porro, in MG producta ſumatur ipſis ME, MG tertia proportionalis MN atque ad alteram partem fiat MH = ME. Denique rectis jungantur puncta AN, AH. Dico factum. Nam (per conſtr.) eſt GM = MC + $\frac{1}{2}$ MC = $\frac{3}{2}$ MC, ideoque GMq = $\frac{3}{2}$ MC × GM. Sed, ob Δ MLC ∽ Δ:lo MLG (pr. 8. l. VI.), erit GM : ML :: ML : MC & MLq = GM × MC; conſequenter GMq = $\frac{3}{2}$ MLq. Quum vero ſit ang. AEM = GLM & angulus GMA communis, erit Δ GML ∽ Δ:lo AEM; ideoque GM : ML :: AM : ME. Sed GM = AM, quia ſunt radii circuli; ergo (pr. 14. l. V.) ML = ME & MLq = MEq. Eſt igitur $\frac{3}{2}$ MEq = ($\frac{3}{2}$ MLq = GMq = AMq =) AEq + EMq (pr. 47. l. I.). Conſequenter AEq = $\frac{1}{2}$ MEq

Prop. 3. Probl. generatim sic solvitur: Sit factum & dic $AM = GM = a$; $M\Sigma = MH = x$, $N\Sigma = z$;

$NM = x + z$; $A\Sigma = \sqrt{a^2 - x^2}$; $AH = \sqrt{A\Sigma^2 + 4x^2} = \sqrt{a^2 + 3x^2}$; $NA = \sqrt{a^2 - x^2 + z^2}$

$NA \left(= \sqrt{a^2 - x^2 + z^2}\right) : AM (= a) :: NM (= x + z) : AH \left(= \sqrt{a^2 + 3x^2}\right)$

$$\begin{array}{l} a^2 - x^2 + z^2 \\ \underline{a^2 + 3x^2} \\ a^4 + 2a^2x^2 + \cancel{a^2z^2} - 3x^4 + 3x^2z^2 = \cancel{a^2x^2} + 2a^2xz + \cancel{a^2z^2} \end{array}$$

$3x^2z^2 - 2a^2xz = 3x^4 - a^2x^2 - a^4$

$9x^2z^2 - 2.3a^2xz + a^4 = 9x^4 - 3a^2x^2 - 2a^4$

$3xz = a^2 \pm \sqrt{9x^4 - 3a^2x^2 - 2a^4}$; $z = \dfrac{a^2 \pm \sqrt{9x^4 - 3a^2x^2 - 2a^4}}{3x}$

$x < a$, $x = > a\sqrt{\frac{2}{3}}$;

Pro $\angle A = 90^\circ$ e $NM^2 = x^2 + 2xz + z^2 = AN^2 + a^2 = 2a^2 - x^2 + z^2$

$x^2 + 2xz = a^2$; $3x^2 - 2a^2 = \mp\sqrt{9x^4 - 3a^2x^2 - 2a^4}$

$\cancel{9x^4} - 12a^2x^2 + 4a^4 = \cancel{9x^4} - 3a^2x^2 - 2a^4$

$6a^4 = 9a^2x^2$; $2a^2 = 3x^2$; $x = a\sqrt{\frac{2}{3}}$

$\frac{3}{2}$ MEq — MEq = $\frac{1}{2}$ MEq. Porro (cfr. pr. 4. l. II.) AHq = (AEq + EHq = AEq + 4 MEq = $\frac{1}{2}$ MEq + 4 MEq) = $\frac{9}{2}$ MEq. Quare AHq $\times$ AEq = $\frac{9}{2}$ MEq $\times$ $\frac{1}{2}$ MEq, atque AH $\times$ AE = ($\frac{3}{2}$ MEq =) AMq. Et quoniam (per constr.) ME : AM (= MG) :: AM : MN, erit (pr. 6. & pr. 17. VI.) non solum $\triangle$ NMA $\backsim$ $\triangle$:lo AEM, sed & MN $\times$ ME = (AMq =) AH $\times$ AE. Ergo MN : AH :: (AE : ME ::) AN : AM, vel AN : AM :: NM : AH. Q. E. F.

Cor. 1. Triangulum quæsitum NAM erit rectangu- #p. 14.
lum ad verticem A. Nam, ut ostendimus, est $\triangle$ NAM $\backsim$ $\triangle$:lo AEM, ideoque ang. NAM = AEM. Sed ang. AEM rectus. Ergo. Invento igitur, ut antea, puncto A in circumferentia circuli GAK, inveniri quoque potest $\triangle$:um NAM, ducendo ad AM rectam perpendicularem AN, rectæ MG productæ in puncto N occurrentem.

Cor. 2. Quia (per demonstr.) AE $\times$ AH = AMq, erit latus AM Trianguli medium proportione inter perpendiculum AE & rectam AH.

Cor. 3. Ob similitudinem Triangulorum NAM, CLM, erit NM : MA :: ML : MC. Sed MC = ($\frac{2}{3}$ MG =) $\frac{2}{3}$ MA; ideoque ME = (ML =) $\frac{2}{3}$ NM. Hinc, data basi NM, facilis elicitur methodus construendi Triangulum desideratum NAM. Abscindatur enim a data basi NM pars tertia NE & a puncto E ducatur recta perpendicularis EA, occurrens peripheriæ circuli super NM tamquam diametro descripti, in puncto A. Junctis AN, AM &c. erit NAM Triangulum desideratum.

Cor. 4. Quoniam AHq = (AEq + 4 EMq = NE $\times$ EM + 8 NE $\times$ EM = 9 NE $\times$ EM =) 9 AEq; erit AH = 3 AE.

Cor. 5. Ex præmissis intelligitur esse NKq = (NMq + MGq + 2 NM $\times$ MG = $\frac{9}{4}$ MEq + MGq + 3 MG $\times$ ME = $\frac{3}{2}$ (MEq + MGq + 2 MG. ME) =) $\frac{3}{2}$ EKq.

Pro-

PROPOSITIO IV. THEOREMA. Fig. 6.

Si ab alterutro Focorum F in Ellipsi ADBE ducantur rectæ ad puncta quævis in peripheria Ellipseos; erit angulorum, quos hæ rectæ faciunt cum tangentibus in iisdem punctis, omnium minimus, quem comprehendit recta FD ducta e foco ad alterutram extremitatum axis minoris D cum contingente Tt, h. e. erit semper ang. $FDT < FNI$.

Dem. Per puncta FDf describatur circulus KFDf. Assumto in ejus peripheria aliquo puncto G, rectis jungantur puncta FG, Gf. Deinde centro D radio DF vel Df describatur circulus FfH. Producatur recta FG donec occurrat peripheriæ circuli FfH in H. Occurrat quoque recta FH Ellipsi in aliquo puncto N. Denique jungantur puncta HD, Hf, Nf. Est igitur, (per pr. 20. l. III. Eucl.) ang. $FDf = 2\,FHf$ & (pr. 21. l. III.) ang. $FDf = FGf$, ideoque ang. $FGf = 2\,FHf$. At (pr. 32. l. I.) ang. $FGf = FHf + GfH$; quare ang. $2\,FHf = FHf + GfH$ & consequenter ang. $FHf = GfH$. Porro, cum sit N punctum in Ellipsi ADBE, erit (per Elem. Sect. Con.) $FN + Nf = FD + Df$. Sed (pr. 20. l. I. Eucl.) $FD + Df$ $(= FD + DH) > FH$, ideoque $FN + Nf > FH$. Auferatur communis FN, fit $Nf > NH$; consequenter (pr. 18. l. I.) ang. $NHf > HfN$. Ergo etiam ang. $GfH > HfN$; addatur ang. GHf, erit $GHf + GfH > GHf + HfN$. Sed ang. $GHf + GfH = FGf$ & ang. $GHf + HfN = FNf$. Quare ang. $FGf > FNf$ vel $FDf > FNf$. Consequenter (cfr. pr. 13. l. I.) ang. $FDT + fDt < FNI + fNi$. Cum autem (p. Elem. Sect. con.) ang. $FDT = fDt$ & ang. $FNI = fNi$, erit ang. $FDT < FNI$ vel ang. $fDt < fNi$.

Similiter ostendetur esse angulum FDT majorem alio quovis angulo FnQ. Ergo angulus FDT (vel fDt) est minimus omnium angulorum, quos faciunt rectæ a

focis

focis ad puncta quævis N vel n ductæ cum rectis Ellipsin iisdem in punctis contingentibus. Q. E. D.

Schol. Vidimus quidem alibi (Cours de Phys. Experim. par D. Desaguliers, T. I. Lec. V. not. 7.) aliam sed paulo proliziorem hujus propositionis demonstrationem. Cum tamen facilius demonstrari possit, nostram utpote valde simplicem hic proferre voluimus.

PROPOSITIO. V.

Invenire integrale æquationis: $m\left(x - \frac{ydx}{dy}\right) = y^n\left(1 \mp \frac{dx^2}{dy^2}\right)^{\frac{n}{2}}$.

Sit $\sqrt{1 \mp \frac{dx^2}{dy^2}} = v$, adeoque in

Cas. 1. Adhibito signo superiori, $\frac{dx}{dy} = \sqrt{1 - v^2}$, seu $dx = dy\sqrt{1 - v^2}$, unde fit $m(x - y\sqrt{1 - v^2}) = y^n v^n$. Quæ æquatio differentiata atque per y & v divisa dat (ob $dx - dy\sqrt{1 - v^2} = 0$), $\frac{mdv}{\sqrt{1 - v^2}} = ny^{n-2} v^{n-1}dy + nv^{n-2}y^{n-1}dv$, & integrando $\int \frac{mdv}{\sqrt{1 - v^2}} = \frac{n}{n-1} \cdot y^{n-1}v^{n-1}$. Cum vero $\int \frac{mdv}{\sqrt{1 - v^2}}$ sit arcus circuli, radio $= 1$, descripti, cujus Sinus $= v = \sqrt{1 - \frac{dx^2}{dy^2}}$; ponatur Sin $Z = v$, adeoque $v^{n-1} = \sin Z^{n-1}$, $\sqrt{1 - v^2} = \text{Cos } Z$, & erit, addita constante, $y^{n-1} = \frac{n-1}{n} \cdot \frac{1}{\text{Sin } Z^{n-1}} \times m(Z + C)$, $y = \frac{1}{\text{Sin } Z}$ √Bm s.p. 20√

$\frac{1}{Sin\,Z}\sqrt[n-1]{\frac{n-1}{n}.\,m\,(Z+C)}$. Quibus ipsorum y, $\sqrt{1-v^2}$, y^n & v^n valoribus substitutis in æquatione: $m\,(x-y\sqrt{1-v})=y^nv^n$, & facta debita reductione reperitur:

$$x=\sqrt[n-1]{\frac{n-1}{n}.\,m\,(Z+C)}\times Cot.\,Z+\overline{\frac{n-1}{n}.\,(Z+C)}$$

seu $x=y\,Sin\,Z\times\overline{Cot.\,Z+\frac{n-1}{n}.\,(Z+C)}$.

Quibus æquationibus continetur relatio inter x & y, quando $m\left(x-\frac{ydx}{dy}\right)=y^n\left(1-\frac{dx^2}{dy^2}\right)^{\frac{n}{2}}$.

Cas. II. Nunc erit $\frac{dx}{dy}=\sqrt{v^2-1}$, seu $dx=dy\sqrt{v^2-1}$, quare, debita facta substitutione, fit $m\,(x-dy\sqrt{v^2-1})=y^nv^n$, ac sumtis differentiis, $m\left(dx-dy\sqrt{v^2-1}-\frac{vydv}{\sqrt{v^2-1}}\right)=ny^{n-1}v^ndy+nv^{n-1}y^ndv$, quæ divisa per y & v atque ob $dx-dy\sqrt{v^2-1}=0$, evadit: $-\frac{mdv}{\sqrt{v^2-1}}=ny^{n-2}v^{n-1}dy+nv^{n-2}y^{n-1}dv$; cujus facta integratione, est $\frac{n}{n-1}.\,y^{n-1}v^{n-1}=\int\frac{-mdv}{\sqrt{v^2-1}}=m$ Log $[a.\,\sqrt{v^2-1}-v)$, denotante a quantitatem quandam constantem. Posito jam $v=\frac{1}{Sin\,Z}$, est $\sqrt{v^2-1}=\sqrt{\frac{1}{Sin\,Z^2}-1}=\frac{1}{Sin\,Z}\sqrt{1-Sin\,Z^2}=\frac{Cos\,Z}{Sin\,Z}$,

$\sqrt{v^2}$

$\sqrt{v^2-1}-v = \frac{Cos\,Z}{Sin\,Z} - \frac{1}{Sin\,Z} = -\frac{1-Cos\,Z}{Sin\,Z} = -$ Tang $\frac{1}{2}$ Z. Qvi valores ipsorum v, $\sqrt{v^2-1}$ & $\sqrt{v^2-1}-v$ si in æquatione $\frac{n}{n-1}\,y^{n-1}v^{n-1} = m\,Log.\,[a.\sqrt{v^2-1}-v]$ substituantur ac debita fiat reductio obtinetur æquatio: $y^{n-1} = \frac{n-1}{n}.\,Sin\,Z^{n-1}.\,m$ Log $[-a.$ Tang $\frac{1}{2}$ Z]; unde $y = Sin\,Z\sqrt[n-1]{\frac{n-1}{n}.\,m\,\text{Log}(-a.\text{Tang}\,\tfrac{1}{2}\,Z)}$. Consequenter his jam inventis valoribus insertis æquationi: $m\,(x-y\sqrt{v^2-1}) = y^n v^n$, eruitur æquatio:

$$m\left(x - Sin\,Z\sqrt[n-1]{\frac{n-1}{n}.\,m\,L.(-a.\,\text{Tang}\,\tfrac{1}{2}\,Z)} \times \frac{Cos\,Z}{Sin\,Z}\right) = \frac{n-1}{n}\,Sin\,Z^n.m\text{Log}(-a.\text{Tang}\,\tfrac{1}{2}Z)\sqrt[n-1]{\frac{n-1}{n}\,m\,L.(-a.\,\text{Tang}\,\tfrac{1}{2}\,Z)} \times \frac{1}{Sin\,Z^n},$$

unde facile invenitur $x = \sqrt[n-1]{\frac{n-1}{n}.\,m\,Log\,(-a.\,\text{Tang}\,\tfrac{1}{2}\,Z)} \times Cos\,Z + \frac{n-1}{n}.\,Log\,(-a.\,\text{Tang}\,\tfrac{1}{2}\,Z)$, seu $x = \frac{y}{Sin\,Z} \times Cos\,Z + \frac{n-1}{n}.\,Log\,(-a.\,Tang\,\tfrac{1}{2}\,Z)$. Hæ æquationes exhibent relationem inter x & y, quando $m\left(x-\frac{y\,dx}{dy}\right) = y^n\left(1+\frac{dx^2}{dy}\right)^n$.

 Aliter.

Aliter. Si $\frac{dx}{dy} = u$, mutatur æquatio: $m\left(x - \frac{y\,dx}{dy}\right) = y^n \left(1 \mp \frac{dx^2}{dy^2}\right)^{\frac{n}{2}}$ in hanc æquationem: $m\,(x - yu) = y^n (1 \mp u^2)^{\frac{n}{2}}$, quæ differentiata & per $y\sqrt{1 \mp u^2}$ divisa, dat, (ob $dx - u\,dy = 0$), $-\frac{m\,du}{\sqrt{1 \mp u^2}} = ny^{n-2}dy\,(1 \mp u^2)^{\frac{n-1}{2}} \mp ny^{n-1}(1 \mp u^2)^{\frac{n-3}{2}}\,u\,du$, adeoque integrando $\int\frac{-m\,du}{\sqrt{1 \mp u^2}} = \frac{n}{n-1}.\ y^{n-1}(1 \mp u^2)^{\frac{n-1}{2}}$. Jam vero $\int\frac{-m\,du}{\sqrt{1 - u^2}}$ = arc. circuli cujus Cosinus $= u$, & $\int\frac{-du}{\sqrt{1 + u^2}} = \log(\sqrt{1 + u^2} - u)$. Consequenter.

Cas. I. Posito $u = Cos\,Z$, ideoque $\sqrt{1 - u^2} = Sin\,Z$, erit $y^{n-1} = m.\ \frac{n-1}{n}.\ \frac{1}{Sin\,Z^{n-1}}.\ (Z + C)$, & $x = y\,Sin\,Z.\ (Cot\,Z + \frac{n-1}{n}.\ (Z + C))$, ut antea.

Cas. II. Sumto $u = Cot\,Z$, erit $\sqrt{1 + u^2} = \frac{1}{Sin\,Z}$ & $\sqrt{1 + u^2} - u = Tang\,\frac{1}{2}\,Z$. Consequenter $y^{n-1} = Sin\,Z^{n-1}.\ \frac{n-1}{n}.\ m\,Log\,(a.\ Tang\,\frac{1}{2}\,Z)$, $x = \frac{y}{Sin\,Z} \times \overline{Cos\,Z + \frac{n-1}{n}.\ Log\,(a.\ Tang\,\frac{1}{2}\,Z)}$; quæ binæ ultimæ æquationes signo tantum constantis ab antea inventis differunt.

Prop. 5 generalius sic solvi potest:

$$m\left(x-\frac{y\,dx}{dy}\right)=y^n\left(1-\frac{a^2dx^2}{dy^2}\right)^{\frac{n}{2}};\quad \frac{a\,dx}{dy}=\mathrm{Cos}\,\varphi;\ \left(1-\frac{a^2dx^2}{dy^2}\right)^{\frac{1}{2}}=\mathrm{Sin}\,\varphi;$$

$$m\left(x-\frac{y\,\mathrm{Cos}\,\varphi}{a}\right)=y^n\,\mathrm{Sin}\,\varphi^n;\quad dx=\frac{\mathrm{Cos}\,\varphi\,dy}{a};$$

$$\frac{m\,y\,\mathrm{Sin}\,\varphi\,d\varphi}{a}=n\,y^{n-1}\,\mathrm{Sin}\,\varphi^n dy+n\,y^n\,\mathrm{Sin}\,\varphi^{n-1}\,\mathrm{Cos}\,\varphi\,d\varphi;\ \frac{n-1}{n\,y\,\mathrm{Sin}\,\varphi}$$

$$\frac{n-1}{n}\cdot\frac{m\,d\varphi}{a}=\overline{n-1}\,y^{n-2}\,\mathrm{Sin}\,\varphi^{n-1}dy+\overline{n-1}\,y^{n-1}\,\mathrm{Sin}\,\varphi^{n-2}\,\mathrm{Cos}\,\varphi\,d\varphi=d\left(y^{n-1}\,\mathrm{Sin}\,\varphi^{n-1}\right)$$

$$y^{n-1}\,\mathrm{Sin}\,\varphi^{n-1}=\frac{\overline{n-1}\,m}{n\,a}(C+\varphi);$$

$$m\left(x-\frac{y\,\mathrm{Cos}\,\varphi}{a}\right)=\frac{\overline{n-1}\,m\,y\,\mathrm{Sin}\,\varphi\,(C+\varphi)}{n\,a}$$

$$x=\frac{y\,\mathrm{Sin}\,\varphi}{a}\left(\mathrm{Cotg}\,\varphi+\frac{n-1}{n}\overline{C+\varphi}\right)$$

Si $n=0$; erit $y^{-1}=\infty$ seu $y=0$; & $x=\frac{0}{0}$

sed ob $x-\frac{y\,dx}{dy}=\frac{1}{m}$; $\frac{dy}{m}=x\,dy-y\,dx$;

$$-\frac{dy}{my^2}=\frac{y\,dx-x\,dy}{y^2};\quad C+\frac{1}{my}=\frac{x}{y};\ x=Cy+\frac{1}{m}\ \text{vel}\ y=\frac{mx+1}{mC};$$

Si $n=1$; erit $y^0=1=0$; & $x=0$;

$$\text{sed}\ -\frac{m\,du}{\sqrt{1\mp u^2}}=\frac{dy}{y}\mp\frac{u\,du}{1\mp u^2}\ \text{ergo}\ L\frac{y\sqrt{1\mp u^2}}{C}=+\int\frac{-m\,du}{\sqrt{1\mp u^2}}$$

Cas. 1°. $u=\mathrm{Cos}\,Z$, $L\frac{y\,\mathrm{Sin}\,Z}{C}=mZ$; $y=\frac{C\,e^{mZ}}{\mathrm{Sin}\,Z}$;

Cas. 2°. $u=\mathrm{Cotg}\,Z$: $\int\frac{-m\,du}{\sqrt{1+u^2}}=m\,\mathrm{Log.}\,\mathrm{tg}\,\tfrac{1}{2}Z$; $y=\frac{C\,\mathrm{tg}\,\tfrac{1}{2}Z^m}{\mathrm{Sin}\,Z}$.

Sign. − pro n impari, sed utrumque pro n pari.

Schol. 1. Posito in Casu 2:o $v = -\frac{1}{Sin\ Z}$, obtinetur $y = \mp Sin\ Z \sqrt[n-1]{\frac{n-1}{n}\, m\ Log\ (a.\ Cot\ \tfrac{1}{2}\ Z)}$, (adhibito signo — vel + prout n est numerus par vel impar) & $x = \frac{y}{Sin\ Z} \times \overline{Cos\ Z - \frac{n-1}{n}.\ Log\ (a.\ Cot\ \tfrac{1}{2}\ Z)}$. Si autem ponatur $u = Tang\ Z$, reperitur $y =$ $Cos\ Z \sqrt[n-1]{\frac{n-1}{n}.\ m\ Log\ (a.\ Tang.\ \overline{45^\circ - \tfrac{1}{2}\ Z})}$ & $x =$ $\frac{y}{Cos\ Z} \times \overline{Sin\ Z + \frac{n-1}{n}\ Log\ (a.\ Tang\ 45^\circ - \tfrac{1}{2}\ Z)}$.

Schol. 2. Æquationem: $m\left(x - \frac{y\,dx}{dy}\right) = y^n \left(1 + \frac{dx}{dy}\right)^{\frac{n}{2}}$, integravit Cel. D:nus Profess. ANDR. PLANMAN in Dissert. sua de Methodo Tangentium inversa, ubi pag. 14 & 15 pro hoc casu exhibet formulas integrales: $\frac{y^{n-1}}{n-1} = -\frac{m}{n.(1+z^2)^{\frac{n-1}{2}}}.\ (Log.\ \overline{z + \sqrt{1+z^2}})$ & $x = \frac{1}{m}\, y^n\, (1+z^2)^{\frac{1}{2}n} + yz$. Quæ, sumto $z = u = Cot\ Z$, facile in illas, quas supra invenimus, commutantur.

Volupe quidem mihi esset Formularum hic propositarum ad casus quosdam speciales facere adplicationem, earumque tentare constructionem; temporis autem angustia impeditus hisce subsistere cogor.

S. D. G.

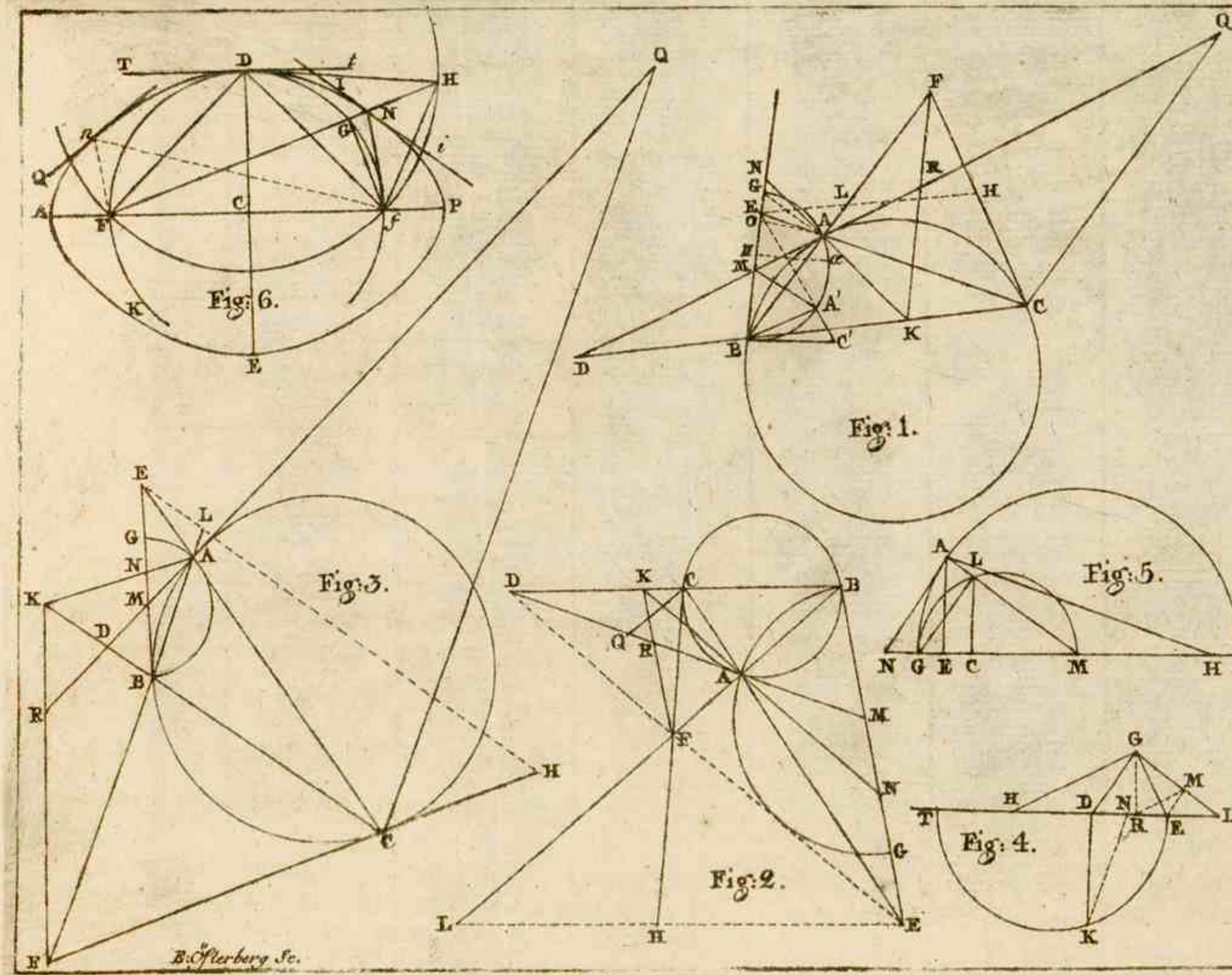
Fig: 6.
Fig: 1.
Fig: 3.
Fig: 5.
Fig: 2.
Fig: 4.
B: Österberg Sc.

EMEND. Pag. 10. lin. 10 — 15 loco: Si enim BC..... absurdum (pr. 13. l. I.). leg. Nam Ang BAC > (ABE =) BCA (Pr. 16. l. I. & 32. l. III.). Pag. 16. lin. 29 loco. majorem, leg. minorem.